Computing the level of a modular rigid Calabi-Yau threefold

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Abstract

In a previous article (cf. [DM]), the modularity of a large class of rigid Calabi-Yau threefolds was established. To make that result more explicit, we recall (and re-prove) a result of Serre giving a bound for the conductor of "integral" 2-dimensional compatible families of Galois representations and apply this result to give an algorithm that determines the level of a modular rigid Calabi-Yau threefold. We apply the algorithm to three examples.

1 Introduction

In [DM], modularity for a large class of rigid Calabi-Yau threefolds defined over $\mathbb{Q}$ was established, by an application of Wiles techniques combined with some solved cases of Serre's conjecture and results on crystalline representations. As other authors have remarked, a drawback of our result is that it does not give a way to determine the corresponding modular form: it is well-known (this is an instance of compatibility with the local Langlands correspondence) that the level of this modular form agrees with the conductor of the compatible family of Galois representations attached to the rigid Calabi-Yau, the problem is that the determination of this conductor is not

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an easy task. So, in order to make our result more useful, in the present note we will describe a simple algorithm that, without any restriction, determines the level of the modular form corresponding to a given modular rigid Calabi-Yau threefold. We will start by recalling a result of Serre (cf. [Se]) giving a universal bound for the exponents of the primes of bad reduction in the conductor of the Galois representations attached to a rigid Calabi-Yau threefold (assuming modularity). In fact, the bound given by Serre is the same holding for elliptic curves defined over $\mathbb{Q}$.

After briefly recalling the ideas behind Serre's proof of this result, we will re-prove it by using congruences between rigid Calabi-Yau threefolds and elliptic curves (or, in the residual reducible case, Hecke characters). With this bound, which is also a bound for the level of the searched modular form, we only have a finite number of modular forms as candidates for a given Calabi-Yau, so by elimination we easily determine the right one. We will illustrate this procedure by determining the right newform for three examples of rigid Calabi-Yau threefolds. In the examples, we use the values of the traces of the images of a few Frobenius elements (these values appear for example in [Y]) and the corresponding eigenvalues of newforms of weight 4 and several levels, most of them available in the tables in W. Stein website [St], and the rest computed with MAGMA.

To speed up the process, we will use in the last example (which involves computations with high levels) mod 5 congruences between weight 4 and weight 2 newforms, so that we can switch to spaces of weight 2 newforms where more tables are available.

Through all this article, we will assume that we are working with a modular rigid Calabi-Yau threefold. Modularity for most of the known examples, and in particular for the three examples that we will consider, follows easily from the main theorem in [DM].

Let us remark that in each known example of rigid Calabi-Yau threefold, both the fact that the variety is modular and the exact value of the level of the corresponding modular form, were also established independently of the results in [DM] by other methods (cf. [Y]). The advantage we see is that with our approach (combining the result in [DM] with the present note) we have a "general result" that gives both modularity (the theoretical result) and the level (the algorithm) for most of the known examples and for many examples to come.

2 The bound for the conductor/level

Let X be a modular rigid Calabi-Yau threefold defined over $\mathbb{Q}$ (for definitions, see [Y]), and let $\{\rho_\ell\}$ be the compatible family of (2-dimensional, continuous, odd, irreducible) Galois representations giving the action of the Galois group of $\mathbb{Q}$ on the ℓ-adic cohomology groups $H^3_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$. Because the family corresponds to a modular form f of weight 4 (whose level N contains only primes of bad reduction of X), the "conductor" c of the family is well-defined: if we take any prime ℓ where X has good reduction, so that ℓ is not in the ramification set of the family $\{\rho_\ell\}$, then c agrees with the prime-to-ℓ part of the conductor of ρ_ℓ, which also agrees with the level N of f.

Remark: More generally, even if ℓ is a prime where the representations ramify, if we take $p \neq \ell$, the p-part of the conductor of the family (equal to the p-part of the level of the corresponding modular form) agrees with the p-part of the conductor of ρ_ℓ.

Let S be the (finite) set of primes of bad reduction of X. For every prime $p \in S$, let e_p be the exponent of p in the level N of f (equal to the exponent of p in the conductor c of the family of Galois representations). Observe that (contrary to what happens in the case of abelian varieties) it is perfectly possible that for some $p \in S$, we have $e_p = 0$. In [Se], section 4.8, Serre gives a bound for these exponents. He assumes the truth of Serre's conjecture in order to ensure that the residual representations, when irreducible (they can only be reducible for finitely many primes), are modular. In our case, we are working with this modularity assumption, therefore the result of Serre applies:

Theorem 2.1 *(J-P. Serre) Let $\{\rho_\ell\}$ be the compatible family of Galois representations attached to a modular rigid Calabi-Yau threefold X with bad reduction set S. Then the conductor c of this family, which agrees with the level N of the corresponding weight 4 modular form, can be bounded as follows: for every prime $p \in S$, the exponent e_p of p in N verifies $e_p \leq 2$ if $p > 3$, $e_p \leq 5$ if $p = 3$, and $e_p \leq 8$ if $p = 2$.*

Remark: For $p = 2$, Serre gives a proof of the weaker inequality $e_2 \leq 9$, but he remarks (cf. [Se], pag. 216) that a more detailed analysis gives $e_2 \leq 8$. In any case, in our proof of the theorem we will prove the bound $e_2 \leq 8$.

Brief description of the original proof: The proof uses the fact that the residual (assume irreducible) representations $\bar{\rho}_\ell$ have image inside $GL_2(\mathbb{F}_\ell)$, and

for a prime $p > 3$, infinitely many of these groups have order prime to p, thus $\bar{\rho}_\ell$ is tamely ramified at p, and this gives the desired bound for the p-part of the conductor of $\bar{\rho}_\ell$ for infinitely many ℓ, and this implies (here is where residual modularity, more precisely the strong version of Serre's conjecture, is used, cf. [Se]) that the same bound holds for the p-part of the modular level N. A similar (though more complicated) argument is used to deal with the cases $p = 2$ and $p = 3$. Here the desired bound is obtained by looking at the (2-part or the 3-part of the) conductor of $\bar{\rho}_\ell$ for primes $\ell \not\equiv \pm 1 \pmod{8}$ or $\ell \not\equiv \pm 1 \pmod 9$ (respectively).

Another proof of Theorem 2.1:
Take $\ell = 5$. We will first be interested in bounding the prime-to-5 part of the conductor c of the family $\{\rho_\ell\}$. To do this, we will bound the conductor of $\bar{\rho}_5$ (taking the definition as in [Se], i.e., considering only the prime-to-5 ramification). Let us divide into two cases:
1) $\bar{\rho}_5$ is reducible: In this case, (semisimplify if necessary), we can assume that $\bar{\rho}_5$ is semisimple, so we have:

$$\bar{\rho}_5 \cong \epsilon \chi^i \oplus \epsilon^{-1} \chi^j$$

where χ is the cyclotomic character. Since $\det(\bar{\rho}_5) = \chi^3$, we have $i + j \equiv 3$ (mod 4). Take $p \neq 5$ and consider the p-part of the conductor of ϵ. Because $\bar{\rho}_5$ is odd, it is well known (irreducible agrees with absolutely irreducible) that the components must also be defined over $\mathbb{F}_5$, so $\text{Image}(\epsilon) \subseteq \mathbb{F}_5^*$. This clearly gives $2^4 = 16$ as a bound for the 2-part of the conductor of ϵ, and $p^1 = p$ as a bound for its p-part for every $p > 2$ ($p \neq 5$). Thus we obtain 2^8 and p^2 (for $p \neq 2, 5$) as bounds for the p-part of the conductor of $\bar{\rho}_5$.
2) $\bar{\rho}_5$ is irreducible: Let $\sigma := \bar{\rho}_5 \otimes \chi$. This representation has determinant equal to χ, then it is known (see [BCDT]) that it is isomorphic to the representation on the 5 torsion of some elliptic curve defined over $\mathbb{Q}$. At any prime $p \neq 5$, the bound for the p-part of the conductor of σ, thus also of $\bar{\rho}_5$, follows from the well-known bound for conductors of elliptic curves (see [Si]).

Now let us compare the conductors of $\bar{\rho}_5$ and ρ_5. Recall that the second of these values agrees with the prime-to-5 part of the conductor of the family $\{\rho_\ell\}$. For a prime $p \neq 5$, it is possible that the exponent e_p' of p in the conductor of $\bar{\rho}_5$ is strictly smaller than the exponent e_p of p in the conductor of ρ_5. However, since the determinant of both representations is unramified

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at p, it is known that $e'_p < e_p$ can only happen in a few particular cases (cf. [C]): $(e'_p = 0, e_p = 2)$; $(e'_p = 0, e_p = 1)$ and $(e'_p = 1, e_p = 2)$ (*).

So, if $e'_p = e_p$, having obtained the right bound for $\bar{\rho}_5$ we also have it for ρ_5, and if $e'_p < e_p$ then $e_p \leq 2$. In any case, we obtain the right bound for the conductor of ρ_5.

As for the 5-part of the conductor of the family $\{\rho_\ell\}$, just observe (as in Serre's proof) that for $\ell = 7$, the order of $\mathrm{GL}_2(\mathbb{F}_7)$ is not multiple of 5, then the 5-part of the conductor of $\bar{\rho}_7$ is at most 5^2, and again using (*) we see that this bound also works for ρ_7.

3 Finding the right newform

With the bound given in Theorem 2.1, we now have a method to find the modular form corresponding to a given modular rigid Calabi-Yau threefold X: Let S be the set of bad reduction primes of X, and let

$$B = \prod_{p \in S} p^{b_p}$$

where the exponents b_p are the bounds given in the theorem. We have to consider all spaces of weight 4 newforms with level N dividing B, and for any newform f in each of these spaces with field of coefficients $\mathbb{Q}_f = \mathbb{Q}$, compare a few eigenvalues a_p with the traces t_p of the images of Frobenius (for $p \notin S$) for the geometric Galois representations attached to the Calabi-Yau threefold. Whenever $a_p \neq t_p$ for a single p, the newform is discarded. With this procedure, by elimination, the (unique) modular form corresponding to X is easily found.

Remark: If f is a newform (with eigenvalues $a_p \in \mathbb{Z}$) not corresponding to X, we should estimate the size of the smallest p such that we have $a_p \neq t_p$. In all computed examples, this always happens for a small p, but for theoretical reasons, let us recall that there is a bound T (Sturm's bound) easily computed in terms of our "maximal possible level B" such that, $a_p = t_p$ for every $p \nmid B, p \leq T$ implies that f does correspond to X. Thus the elimination procedure necessarily finishes at a prime p smaller than T.

Incidentally, observe that this gives an alternative way of determining the right newform f: if you suspect which is the right f, instead of eliminating the other candidates, just check the equality $a_p = t_p$ up to Sturm's bound T. This suffices for a proof. This method is not practical because since B can

be large, the bound T sometimes becomes too large for computations.

3.1 The Examples

First Example: Let X_1 be the rigid Calabi-Yau with bad reduction only at 2 constructed by Werner and van Geemen (cf. [Y]), with the following values for t_p ($p \leq 7$, $p \neq 2$): $-4, -2, 24$.
Since it has good reduction at 3 and 7, it is modular (cf. [DM]). We know from Theorem 2.1 that the corresponding modular form has level dividing 256, and comparing the first eigenvalues of all newforms of such levels with the values of t_p listed above, we conclude that the modular form f_1 corresponding to X_1 has level 8.

Second Example: Let X_2 be the rigid Calabi-Yau with bad reduction only at 5 constructed by Schoen (cf. [Se]). Again, the main theorem of [DM] implies that it is modular, and Theorem 2.1 gives us 25 as a bound for the level of the corresponding modular form. Using only the values of t_2 and t_3 we conclude that it corresponds to a newform of level 25.

Third Example: Let X_3 be the rigid Calabi-Yau with bad reduction at 2 and 5 constructed by Werner and van Geemen (cf. [Y]), with the following values for t_p ($p = 3, 7, 11, 13, 17$ and 19): $-2, -26, -28, -12, 64, -60$ (*).
Again [DM] gives modularity. Theorem 2.1 gives a large bound for the level of the corresponding newform: $B = 256 * 25 = 6400$. To speed up the process of elimination, we have applied a different trick to cases of large level. We have divided in two cases:
a) level N divisible by 16: In this case, the trick is the following: consider the mod 5 representation $\bar{\rho}_5$, the first traces of this representation are the reductions mod 5 of the values t_p listed in (*). Observe that the hypothesis $16 \mid N$ implies that the conductor of $\bar{\rho}_5$ is also divisible by 16 (as in the previous section, cf. [C]), and this in turn implies that $\bar{\rho}_5$ must be irreducible, since it is not hard to see from the values of a few t_p (reduced mod 5) that if it were reducible it (in fact, its semisimplification) would be unramified at 2. Now consider the twisted representation $\sigma := \bar{\rho}_5 \otimes \chi$. This irreducible modular representation must correspond to a weight 2 newform, whose level divides 6400 and is multiple of 16, and whose first eigenvalues a_p should agree modulo 5 with $p \cdot t_p$, thus the value of these eigenvalues a_p modulo 5 should

be (for $p = 3, 7, 11, 13, 17$ and 19):

$$-1, -2, -3, -1, -2, 0 \qquad (**)$$

We search through all these spaces of newforms (for all newforms up to level 3200, and also for those of level 6400 with $\mathbb{Q}_f = \mathbb{Q}$, the eigenvalues are listed in the tables in [St], for the remaining newforms of weight 2 and level 6400, we performed computations with MAGMA). We eliminate all newforms such that $\mathbb{Q}_f \neq \mathbb{Q}$ and there is no prime above 5 of residue class degree 1. For the remaining newforms, in most cases the values of a_3 and a_7 modulo 5 already do not match with (**), and finally using the other values in (**) we eliminate all newforms. We conclude that it is impossible that the conductor of ρ_5 be multiple of 16, thus we have $16 \nmid N$.

b) level N not divisible by 16: Having discarded case a), we know that the 2-part of the conductor is at most 8, and comparing the first values of t_p listed in (*) with all newforms of weight 4 and level dividing $8 * 25 = 200$, the only one that matches is a newform of level 50. Thus we conclude that the Calabi-Yau threefold X_3 is modular of level 50.

Final Remark: Assuming that $\bar{\rho}_5$ is irreducible, after twisting it by χ we obtain the representation σ that must correspond to some newform of weight 2 and level dividing 50. But the only such newform (with its first eigenvalues modulo 5 as in (**)) corresponds to an elliptic curve of conductor 50, and it is known that this elliptic curve has a rational 5-torsion point, contradicting the irreducibility of $\bar{\rho}_5$. We conclude that $\bar{\rho}_5$ is reducible.

4 Final Remark

It follows from recent results of Taylor that the compatible family of Galois representations attached to any rigid Calabi-Yau threefold (modular or not) is "strongly compatible" (cf. [T]). This strong compatibility implies that the conductor of the family is well-defined (as in the case of Galois representations attached to modular forms, recall the discussion in section 2). In the proof of theorem 2.1 given in this note, the Calabi-Yau threefold was assumed to be modular only to apply this "independence of ℓ" of the conductor, thus we conclude that the bound for the conductor given in theorem 2.1 is true for any rigid Calabi-Yau threefold (modular or not).

5 Bibliography

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